

ON THE SUITABILITY OF FIRST-ORDER DIFFERENTIAL MODELS FOR TWO-PHASE FLOW PREDICTION

A. V. JONES

Commission of the European Communities Joint Research Centre, 21020 Ispra (Va), Italy

A. PROSPERETTI

Dipartimento di Fisica, Università di Milano, Via Celoria 16, 20133 Milano, Italy

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Abstract—The stability features of a general class of one-dimensional two-phase flow models are examined. This class of models is characterized by the presence of first-order derivatives and algebraic functions of the flow variables, higher-order differential terms being absent, and can accommodate a variety of physical effects such as added mass and unequal phase pressures in some formulations. By taking a general standpoint, a number of results are obtained applicable to the entire class of models considered. In particular, it is found that, despite the presence of algebraic terms in the equations (describing, e.g. drag effects) the stability criteria are independent of the wavenumber of the perturbation. As a consequence, reality of characteristics is necessary, although not sufficient, for stability. To illustrate the theory, three specific models are considered in detail.

INTRODUCTION

In this paper the stability features of a general class of one-dimensional two-phase flow models are examined. The salient characteristic of this general class is that it contains only first-order differential and algebraic terms. Several specific two-phase flow models available in the literature are of this type. Stuhmiller (1977) was led to such a model by the consideration of added mass and dynamic pressure in the context of bubbly flow. Ardron (1980), Rousseau & Ferch (1979), and Banerjee & Chan (1979) considered two-pressure models motivated by the effect of gravity on stratified flows and showed that the difference in pressure was proportional to the gradient of the void fraction. Models belonging to this class are also currently used in large nuclear safety codes such as RELAP-5 (Ransom 1980). Even for models containing higher-order derivatives such as those of Ramshaw & Trapp (1978), Arai (1980), and Banerjee (1980), the stability at long wavelength is dominated by the first-order differential terms (Prosperetti & Jones 1985) and therefore in this sense they fall within the class considered here.

It is not our purpose here to add a new specific model to an already crowded list. Rather, by taking a more general standpoint, we are able to obtain a number of results applicable to the entire class of models considered. In this way some criteria for the evaluation of new and existing models can be formulated concerning steady flows, their stability, the reality of characteristics, and other features. For example, an interesting result that we find is that in spite of the presence of algebraic terms the stability properties of steady uniform flows are independent of the wavelength of the perturbation. This circumstance is a clear indication of the incompleteness of this class of models and shows, for example, that phenomena such as flow regime transitions require the presence of higher-order derivatives in the equations.

As examples of the possible use of the results obtained three specific models are considered in detail. Interestingly the simplest such model, which is in widespread use in engineering calculations, is found to possess a number of unphysical features. The other models contain two different formulations of added mass effects, and while one of them has acceptable stability properties, the second one appears to be incomplete and to require the addition of further terms.

In the course of our analysis we make a number of simplifying assumptions, the most significant of which is the neglect of the compressibility of the individual phases. This has the

effect of rendering the consideration of the energy equations superfluous. Although our results cannot therefore be applied to situations for which compressibility effects are important (e.g. pressure wave propagation in bubbly flows), there remains a number of practical situations in which they are relevant, such as droplet flow at low velocity and liquid-liquid and fluid-solid flows whenever extremes of pressure are absent.

A GENERAL CLASS OF TWO-PHASE MODELS

The general class of one-dimensional, adiabatic, two-phase models to be considered in this paper is described by the following equations:

$$\frac{\partial}{\partial t} (\alpha_G \rho_G) + \frac{\partial}{\partial x} (\alpha_G \rho_G V_G) = 0, \quad [1a]$$

$$\frac{\partial}{\partial t} (\alpha_L \rho_L) + \frac{\partial}{\partial x} (\alpha_L \rho_L V_L) = 0, \quad [1b]$$

$$\begin{aligned} \frac{\partial}{\partial t} (\alpha_G \rho_G V_G) + \frac{\partial}{\partial x} (\alpha_G \rho_G V_G^2) + \alpha_G \frac{\partial p}{\partial x} - \alpha_G \rho_G A_G \\ = \sum_{j=G,L} \left(h_{Gj} \frac{\partial V_j}{\partial t} + k_{Gj} \frac{\partial V_j}{\partial x} \right) + m_G \frac{\partial \alpha_G}{\partial t} + n_G \frac{\partial \alpha_G}{\partial x}, \end{aligned} \quad [2a]$$

$$\begin{aligned} \frac{\partial}{\partial t} (\alpha_L \rho_L V_L) + \frac{\partial}{\partial x} (\alpha_L \rho_L V_L^2) + \alpha_L \frac{\partial p}{\partial x} - \alpha_L \rho_L A_L \\ = \sum_{j=G,L} \left(h_{Lj} \frac{\partial V_j}{\partial t} + k_{Lj} \frac{\partial V_j}{\partial x} \right) + m_L \frac{\partial \alpha_L}{\partial t} + n_L \frac{\partial \alpha_L}{\partial x}. \end{aligned} \quad [2b]$$

Here ρ , V and p indicate density, velocity and pressure and the indices G and L distinguish the two phases. The volume fractions α_L , α_G are related by

$$\alpha_L + \alpha_G = 1. \quad [3]$$

Equations [1] express conservation of mass of each phase and implicitly assume that no mass transfer occurs between the phases. The second pair of equations is a statement of the momentum balance for each phase.

The present model is restricted to contain only the derivative terms which explicitly appear in [2]. The quantities A_i , h_{ij} , k_{ij} , m_i , n_i are therefore taken to be algebraic functions of the flow variables V_{LG} , ρ_{LG} , α_{LG} . For the following developments we must exclude pressure from this list. A number of models conform to this restriction (Wallis 1969, Harlow & Amsden 1975, Stewart 1979, Liles & Mahaffy 1979, Spalding 1979, Smith 1980) especially for the disperse flow regime (Prosperetti & Jones 1984).

Although a single pressure appears in the momentum equations [2], unequal-pressure models can be cast in the form of [2] provided that the difference between the phase pressures can be written as an algebraic function of the flow variables as, for instance, in some separated flow models (Rousseau & Ferch 1979, Ardron 1980, Banerjee 1980). In addition to accounting for possible pressure differences, the terms on the right-hand side of [2] can describe a variety of physical effects such as added mass, unsteady contributions to the drag, correlation contributions arising from averaging of the exact conservation equations, and interparticle forces. The algebraic functions A_{LG} are intended to model steady drag and body forces. Although our results are not dependent on any specific structure for these functions we may quote as an example the form used by Harlow & Amsden (1975):

$$A_G = g + \frac{H(V_L - V_G) - K_G V_G}{\alpha_G \rho_G} \quad [4a]$$

$$A_L = g + \frac{H(V_G - V_L) - K_L V_L}{\alpha_L \rho_L} \quad [4b]$$

Here g is the component of gravitational acceleration in the direction of the flow, H is the drag coefficient between the phases and K_G and K_L are drag coefficients of the gas and liquid phase with any structure which may be present, e.g. pipe walls.

In order to make the model [1], [2] amenable to an analytical treatment we need to assume that the phase densities $\rho_{L,G}$ are constant. Certainly this is a good approximation for liquid-liquid and liquid-solid flows under most conditions. For gas-solid or gas-liquid flows this assumption is somewhat more restrictive especially in view of the small velocity of pressure waves in bubbly liquids, but still includes a range of practically interesting situations.

With this assumption $\rho_{L,G}$ can be omitted in [1] which can be combined using [3] to give

$$\frac{\partial}{\partial x} (\alpha_L V_L + \alpha_G V_G) = 0, \quad [5]$$

which shows the space independence of the volume velocity U given by

$$U = \alpha_L V_L + \alpha_G V_G. \quad [6]$$

In terms of this quantity, using [3], the volume fractions can be expressed as

$$\alpha_G = \frac{U - V_L}{V_G - V_L}, \quad \alpha_L = \frac{U - V_G}{V_L - V_G}. \quad [7]$$

Note that, since $\alpha_{G,L} \geq 0$, these relations show that U lies between V_G and V_L .

Using these relations the terms involving derivatives of $\alpha_{L,G}$ on the rhs of [2] can be expressed as linear combinations of $\dot{U} = dU/dt$, and time and space derivatives of $V_{G,L}$. The momentum equations can then be written in the form

$$\frac{\partial}{\partial t} (\alpha_i V_i) + \frac{\partial}{\partial x} (\alpha_i V_i^2) + \frac{\alpha_i}{\rho_i} \frac{\partial p}{\partial x} = \alpha_i A_i + \alpha_i \sum_{j=G,L} \left(q_{ij} \frac{\partial V_j}{\partial t} + r_{ij} \frac{\partial V_j}{\partial x} \right) + \alpha_i s_i \dot{U}, \quad i = G, L. \quad [8]$$

The q_{ij} , r_{ij} , s_i are algebraic functions of U and $V_{G,L}$ which can be expressed in terms of h_{ij} , k_{ij} , m_i , n_i in a straightforward manner which we do not need to specify. If [8], written for $i = L$ and $i = G$, are now added, by [6] and [7] one readily finds that

$$\frac{1}{\bar{\rho}} \frac{\partial p}{\partial x} + \dot{U} + (U - V_L) \frac{\partial V_G}{\partial x} + (U - V_G) \frac{\partial V_L}{\partial x} = (\text{rhs})_G + (\text{rhs})_L, \quad [9]$$

where

$$\bar{\rho}^{-1} = \frac{\alpha_L}{\rho_L} + \frac{\alpha_G}{\rho_G} \quad [10]$$

and $(\text{rhs})_i$, $i = G, L$, represents the right-hand side of [8]. Equation [9] can now be used to eliminate $\partial p/\partial x$ from [2] with the result

$$(1 - \alpha_L \eta_G) \frac{\partial V_G}{\partial t} - \alpha_L \eta_L \frac{\partial V_L}{\partial t} + W_{GG} \frac{\partial V_G}{\partial x} + W_{GL} \frac{\partial V_L}{\partial x} = B_G + S_G \dot{U}, \quad [11a]$$

$$-\alpha_G \eta_G \frac{\partial V_G}{\partial t} + (1 + \alpha_G \eta_L) \frac{\partial V_L}{\partial t} + W_{LG} \frac{\partial V_G}{\partial x} + W_{LL} \frac{\partial V_L}{\partial x} = B_L + S_L \dot{U} \quad [11b]$$

in which some auxiliary quantities have been introduced. These are defined as follows:

$$W = \left(\frac{\alpha_L}{\rho_L} V_G + \frac{\alpha_G}{\rho_G} V_L \right) \tilde{p}, \quad [12]$$

$$\eta_G = \left(\frac{q_{LG}}{\rho_G} - \frac{q_{GG}}{\rho_L} \right) \tilde{p}, \quad [13]$$

$$W_{GG} = W + \alpha_L \theta_G, \quad [14]$$

$$W_{GL} = \alpha_L \frac{\tilde{p}}{\rho_G} (V_G - V_L) - \alpha_L \theta_L = \frac{\alpha_L}{\alpha_G} (V_G - W_{LL}), \quad [15]$$

$$\theta_G = \left(\frac{r_{LG}}{\rho_G} - \frac{r_{GG}}{\rho_L} \right) \tilde{p}, \quad [16]$$

with η_L , θ_L , W_{LL} , W_{LG} defined by similar expressions with the indices L and G interchanged. The source terms $B_{G,L}$, $S_{G,L}$ are defined by

$$B_G = \alpha_L \tilde{p} \left(\frac{A_G}{\rho_L} - \frac{A_L}{\rho_G} \right), \quad [17]$$

$$S_G = \frac{\tilde{p}}{\rho_G} + \alpha_L \tilde{p} \left(\frac{s_G}{\rho_L} - \frac{s_L}{\rho_G} \right). \quad [18]$$

Equations [11] together with [7] define a closed system, whose stability properties are investigated in the subsequent sections.

STEADY FLOW

The simplest situation to which [11] can be applied is that of steady flow in a straight pipe. In this case $\partial V_{G,L}/\partial t = 0$, $\dot{U} = 0$, and [11] may be solved for $\partial V_{G,L}/\partial x$ to give

$$\frac{\partial V_G}{\partial x} = - \frac{A_L/\rho_G - A_G/\rho_L}{\frac{\alpha_G}{\rho_G} V_L^2 + \frac{\alpha_L}{\rho_L} V_G^2 + \frac{\alpha_L \theta_G}{\tilde{p}} V_G + \frac{\alpha_G \theta_L}{\tilde{p}} V_L} \alpha_L V_G, \quad [19a]$$

$$\frac{\partial V_L}{\partial x} = \frac{A_L/\rho_G - A_G/\rho_L}{\frac{\alpha_G}{\rho_G} V_L^2 + \frac{\alpha_L}{\rho_L} V_G^2 + \frac{\alpha_L \theta_G}{\tilde{p}} V_G + \frac{\alpha_G \theta_L}{\tilde{p}} V_L} \alpha_G V_L. \quad [19b]$$

For $V_{G,L} \neq 0$ these derivatives are seen to vanish together and to have opposite signs if V_G and V_L have the same sign. Dividing [19b] by [19a] we find that

$$\frac{dV_G}{dV_L} = - \frac{\alpha_L V_G}{\alpha_G V_L} = \frac{U - V_G V_G}{U - V_L V_L}, \quad [20]$$

which can readily be integrated to obtain

$$V_G = \frac{(C-1)V_L}{CV_L - U} U, \quad [21]$$

where the constant C is determined by the boundary conditions at, say, $x = 0$:

$$C = \frac{U}{\alpha_L(0)V_L(0)} = 1 + \frac{\alpha_G(0)V_G(0)}{\alpha_L(0)V_L(0)} = 1 + \frac{\alpha_G V_G}{\alpha_L V_L}. \quad [22]$$

If $V_{G,L}(0) \geq 0$, $C \geq 1$. Note that, since in the steady case [1] implies the spatial independence of the volume velocities $\alpha_G V_G$, and $\alpha_L V_L$, from [22] it is seen that if $CV_L > 0$, V_L maintains the same sign along the entire flow.

In the phase plane (V_G, V_L) of the system [19], [22] represents a relationship between V_G and V_L which must hold at any position along the pipe. Mathematically it corresponds to a family of hyperbolae with asymptotes at $V_L = U/C$ and $V_G = (C - 1)U/C$. All the members of this family pass through the point $V_G = V_L = U$. As the flow evolves along the pipe the point representing the solution of the system [19] moves along a specific hyperbola of the family.

The asymptotic behaviour of the solutions of [19] is determined by the critical points of the system, i.e. the values of V_G and V_L which render the rhs of [19] zero simultaneously. The only such points at a finite distance from the origin in the (V_G, V_L) plane are such that

$$\rho_L A_L = \rho_G A_G, \quad [23a]$$

and points at which the denominator in [19] becomes infinite. However, this latter possibility implies a singular behaviour of the coefficients of the original equations [2] which would represent a breakdown of the model itself. A further possibility which must be dismissed is that singular points might exist for infinite velocities. Indeed, such points would have to belong to the hyperbolae [21] and therefore one of the velocities would be finite and consequently the slip infinite. It is obvious from [2] that [23a] does describe steady uniform flow and indeed $\rho_L A_L$ or $\rho_G A_G$ equals the common value of the pressure gradient in these conditions,

$$\frac{\partial p}{\partial x} = \rho_L A_L = \rho_G A_G. \quad [23b]$$

In the phase plane (V_G, V_L) [23a] represents a curve and the critical points lie at its intersections with the particular hyperbola of the family [21] corresponding to the inlet conditions of the flow. It is well known from the theory of differential equations (Coddington & Levinson 1955) that the stable critical points are the asymptotic values of V_L and V_G as $x \rightarrow \infty$. We are thus led to investigate the stability of such points, which correspond to flows which are not only steady, but also uniform.

STABILITY OF UNIFORM STEADY FLOW

Consider a steady uniform flow with velocities $\bar{V}_L, \bar{V}_G, \bar{U}$. Then equations [23] imply the vanishing of $B_{G,L}$ defined by [17]. To investigate the stability of this flow against small perturbations we set

$$V_{G,L} = \bar{V}_{G,L} + v'_{G,L}, \quad [24]$$

$$U = \bar{U} + u(t). \quad [25]$$

Note that, according to [5], the perturbation $u(t)$ of the volume velocity depends only on the time. Upon insertion of [24] and [25] into [11] and linearization one finds

$$(1 + \alpha_L \eta_G) \frac{\partial v'_G}{\partial t} - \alpha_L \eta_L \frac{\partial v'_L}{\partial t} + W_{GG} \frac{\partial v'_G}{\partial x} + W_{GL} \frac{\partial v'_L}{\partial x} = \beta_{GL} v'_L + \beta_{GG} v'_G + S_G \dot{u} + \frac{\partial B_G}{\partial U} u, \quad [26a]$$

$$(1 + \alpha_G \eta_L) \frac{\partial v'_L}{\partial t} - \alpha_G \eta_G \frac{\partial v'_G}{\partial t} + W_{LL} \frac{\partial v'_L}{\partial x} + W_{LG} \frac{\partial v'_G}{\partial x} = \beta_{LG} v'_G + \beta_{LL} v'_L + S_L \dot{u} + \frac{\partial B_L}{\partial U} u, \quad [26b]$$

where overbars have been dropped for convenience and

$$\beta_{ij} = \frac{\partial B_i}{\partial V_j}, \quad i, j = L, G. \quad [27]$$

It can be verified by direct substitution that the following change of dependent variables:

$$v'_G = v_G + \beta_{GL}c_1(t) + \beta_{GG}c_2(t), \quad [28a]$$

$$v'_L = v_L - \beta_{GG}c_1(t) + \beta_{LG}c_2(t), \quad [28b]$$

where

$$c_1(t) = c_{10} - \frac{u(t) - u(0)}{\alpha_L(\beta_{LL} + \beta_{LG})},$$

$$c_2(t) = \exp [(\beta_{GG} + \beta_{LL})t]$$

$$\times \left\{ c_{20} + \int_0^t \frac{[\beta_{LG}(S_G + \eta_L) + \beta_{LL}(S_L + \eta_G)]\dot{u} + \frac{\partial B_L}{\partial U}(\beta_{GG} + \beta_{LL})u}{\beta_{LG}(\beta_{LL} + \beta_{GG})(1 + \alpha_G\eta_L + \alpha_L\eta_G)} \times \exp [-(\beta_{GG} + \beta_{LL})t'] dt' \right\},$$

and c_{10}, c_{20} are arbitrary constants, reduces [26] to

$$(1 + \alpha_L\eta_G) \frac{\partial v_G}{\partial t} - \alpha_L\eta_L \frac{\partial v_L}{\partial t} + W_{GG} \frac{\partial v_G}{\partial x} + W_{GL} \frac{\partial v_L}{\partial x} = \beta_{GL}v_L + \beta_{GG}v_G, \quad [29a]$$

$$(1 + \alpha_G\eta_L) \frac{\partial v_L}{\partial t} - \alpha_G\eta_G \frac{\partial v_G}{\partial t} + W_{LL} \frac{\partial v_L}{\partial x} + W_{LG} \frac{\partial v_G}{\partial x} = \beta_{LG}v_G + \beta_{LL}v_L. \quad [29b]$$

The solution of this system can be found by separation of variables writing

$$v_{G,L} = \mathcal{U}_{G,L} \exp [ikx + k(c - iK)t], \quad [30]$$

in which $\mathcal{U}_{G,L}$ are constants, k is the wavenumber of the perturbation, and the real constant K , introduced for convenience, is defined by [37] below.

The complex eigenvalues c are determined by substitution of [30] into [29] to find

$$Ak^2c^2 - [ik(2KA - E) + B]kc + k^2F + ik(KB - D) + \beta_{GG}\beta_{LL} - \beta_{GL}\beta_{LG} = 0, \quad [31]$$

where

$$A = 1 + \alpha_L\eta_G + \alpha_G\eta_L, \quad [32]$$

$$B = \beta_{LL} + \beta_{GG} + \eta_G(\alpha_L\beta_{LL} + \alpha_G\beta_{GL}) + \eta_L(\alpha_G\beta_{GG} + \alpha_L\beta_{LG}), \quad [33]$$

$$D = \beta_{LL}(W + \alpha_L\theta_G) + \beta_{GG}(W + \alpha_G\theta_L) \quad [34]$$

$$+ \beta_{LG} \left[\frac{\tilde{p}}{\rho_G} (U - V_G) + \alpha_L\theta_L \right] + \beta_{GL} \left[\frac{\tilde{p}}{\rho_L} (U - V_L) + \alpha_G\theta_G \right],$$

$$E = (1 + \alpha_L\eta_G)(W + \alpha_G\theta_L) + (1 + \alpha_G\eta_L)(W + \alpha_L\theta_G) \quad [35]$$

$$- \alpha_L\eta_L \left[\frac{\tilde{p}}{\rho_L} (U - V_L) + \alpha_G\theta_G \right] - \alpha_G\eta_G \left[\frac{\tilde{p}}{\rho_G} (U - V_G) + \alpha_L\theta_L \right],$$

$$F = K^2A - (W + \alpha_L\theta_G)(W + \alpha_G\theta_L) + \left[\frac{\tilde{p}}{\rho_L} (U - V_L) + \alpha_G\theta_G \right] \left[\frac{\tilde{p}}{\rho_G} (U - V_G) + \alpha_L\theta_L \right]. \quad [36]$$

In order to make the coefficient of c in [31] real we choose

$$K = E/2A. \quad [37]$$

Since k in [30] is real the steady uniform flow under investigation is unstable unless $\text{Re}c \leq 0$. In general, given the equation

$$X^2 + pX + q + ir = 0 \quad [38]$$

with p, q, r real quantities, it is a simple matter to show that the conditions ensuring $\text{Re}X \leq 0$ are

$$p^2q \geq r^2, \quad p \geq 0. \quad [39]$$

The first of these stability conditions applied to [31] gives

$$\frac{B^2}{A} (k^2F + \beta_{GG}\beta_{LL} - \beta_{GL}\beta_{LG}) \geq k^2(KB - D)^2. \quad [40]$$

Equation [40] seems to imply a dependence of the stability upon the wavenumber of the perturbation, as expected from the presence of nondifferential terms in [2]. However no such dependence is present here due to the following relations

$$\alpha_L\beta_{LL} + \alpha_G\beta_{GL} = \alpha_G\beta_{GG} + \alpha_L\beta_{LG} = 0, \quad [41]$$

which imply that

$$\beta_{GG}\beta_{LL} - \beta_{LG}\beta_{GL} = 0. \quad [42]$$

Equation [41] follow readily from the definition [17] of $B_{G,L}$ and [23] which are valid for steady uniform flow. Equation [40], therefore, reduces to

$$\frac{B^2}{A} F \geq (KB - D)^2. \quad [43]$$

The second stability condition derived from [39] is simply

$$\frac{\beta_{GG} + \beta_{LL}}{1 + \alpha_L\eta_G + \alpha_G\eta_L} \leq 0, \quad [44]$$

which is also independent of the wavenumber.

The independence of these two stability conditions from the wavenumber is certainly unrealistic, and indicates that models which can be cast in the form [1], [2] have at best limited validity. Condition [43] will be discussed further in the following sections.

As for the condition [44], we note that

$$\beta_{GG} + \beta_{LL} = \frac{1}{\rho_L} \left(\alpha_L \frac{\partial A_G}{\partial V_G} - \alpha_G \frac{\partial A_G}{\partial V_L} \right) + \frac{1}{\rho_G} \left(\alpha_G \frac{\partial A_L}{\partial V_L} - \alpha_L \frac{\partial A_L}{\partial V_G} \right). \quad [45]$$

Aside from the gravity term g , which does not contribute to the derivatives because it necessarily enters additively in $A_{G,L}$, the $A_{G,L}$ represent the algebraic contributions to the drag forces on the phases, as is clear from their position in [2]. A sufficient condition for

$$\beta_{GG} + \beta_{LL} \leq 0 \quad [46]$$

is that the algebraic drag A_G decreases with V_G and increases with V_L , while the converse holds for A_L . These conditions are not unreasonable away from flow regime transitions; for example, for the simple drag functions [4], the condition [46] reduces to

$$H + \alpha_L^2 \frac{\partial(K_G V_G)}{\partial V_G} + \alpha_G^2 \frac{\partial(K_L V_L)}{\partial V_L} \geq 0. \quad [47]$$

The interphase drag coefficient H is necessarily positive, and the structure drag forces can be expected to be increasing functions of their respective velocities in most cases. The condition [46] is equivalent to [44] provided that

$$1 + \alpha_L \eta_G + \alpha_G \eta_L \geq 0. \quad [48]$$

Note that $\eta_{G,L}$ vanish with the right-hand sides of [2]. Since the left-hand sides of [2] are themselves presumably a fair approximation to momentum balances for the phases, we do not expect the weight of the terms on the right of [2] to be such as to make [48] invalid. Equation [48] will be found to be satisfied in the examples to be analysed below.

We note that use of the definitions [32]–[37] allows an alternative and more explicit form to be found for the first stability condition [43], namely

$$\frac{V_G - V_L}{1 + \alpha_L \eta_G + \alpha_G \eta_L} \cdot [(\beta_{GG} + \beta_{LL})(\alpha_G \beta_{GG} \theta_L - \alpha_L \beta_{LL} \theta_G) + (\beta_{LL} V_L + \beta_{GG} V_G)(\alpha_L \beta_{LL} \eta_G - \alpha_G \beta_{GG} \eta_L)] \geq \frac{\bar{\rho}(V_G - V_L)^2 \left(\frac{\alpha_G}{\rho_G} \beta_{GG}^2 + \frac{\alpha_L}{\rho_L} \beta_{LL}^2 \right)}{1 + \alpha_L \eta_G + \alpha_G \eta_L}. \quad [49]$$

Obviously, this condition is satisfied if $V_G = V_L$.

REALITY OF CHARACTERISTICS AND STABILITY

It is possible to establish an explicit connection between the stability criterion [43] for steady uniform flows and reality of characteristics of the system [11]. The characteristics of this system, $dx/dt = \mu_{\pm}$, are given by

$$\mu_{\pm} = \frac{1}{2} K \pm \left(\frac{F}{A} \right)^{1/2}. \quad [50]$$

These are of course also two of the four characteristics of the original system [1], [2]. The remaining two are found to be infinite because of the assumption that both phases are incompressible. The condition for the reality of the characteristics [50] is $F/A \geq 0$ which is clearly a necessary condition for the stability criterion [43] to be satisfied. We therefore conclude that, for the class of first-order models considered here, reality of characteristics is a necessary condition for the stability of steady uniform flow. This result is not trivial since for differential systems containing algebraic source terms, such as $A_{G,L}$ in [2], reality of characteristics and stability coincide only in the limit $k \rightarrow \infty$ (Ramshaw & Trapp 1978). The present result is a consequence of the independence of the stability criteria [43], [44] on the wavenumber. For the class of models considered in this paper, complex characteristics are bound to result in an unstable flow, although instabilities can also be modelled in the framework of a totally hyperbolic system when F/A is positive but conditions [43] or [44]

are not satisfied. The reality condition above may be written in the more explicit form

$$\frac{1}{4} [\alpha_G(\theta_L - \eta_L V_L) + \alpha_L(\theta_G - \eta_G V_G)]^2 - \tilde{\rho}^2 \frac{\alpha_G \alpha_L}{\rho_G \rho_L} (V_G - V_L)^2 + \alpha_G \alpha_L (V_G - V_L) \left[\tilde{\rho} \left(\frac{\theta_L - \eta_L V_G}{\rho_L} - \frac{\theta_G - \eta_G V_L}{\rho_G} \right) + \theta_L \eta_G - \theta_G \eta_L \right] \geq 0. \quad [51]$$

This condition is clearly met when $V_G = V_L$ provided $\theta_{G,L}, \eta_{G,L}$ are finite in this limit.

In the following sections we shall consider three specific models in the light of the general results obtained above.

A SIMPLE MODEL

We first consider perhaps the simplest model of the class [2], in which the right sides vanish. This model seems to have been first published by Wallis (1969) and has subsequently been extensively used in numerical applications by Harlow & Amsden (1975) and others (e.g. Stewart 1979, Liles & Mahaffy 1979, Spalding 1979, Smith 1980). The two stability conditions [43], [44] reduce in this case to

$$(V_G - V_L)^2 \leq 0, \quad [52]$$

and

$$\beta_{GG} + \beta_{LL} \leq 0. \quad [53]$$

The first relation can only be satisfied for equal velocities, while to the second one the considerations already made regarding the $A_{G,L}$ apply. The characteristics of this model are found from [50] to be

$$\mu_{\pm} = \tilde{\rho} \left[\frac{\alpha_G}{\rho_G} V_L + \frac{\alpha_L}{\rho_L} V_G \pm i \left(\frac{\alpha_G \alpha_L}{\rho_G \rho_L} \right)^{1/2} (V_G - V_L) \right], \quad [54]$$

and are complex unless $V_G = V_L$, as expected from the previous section. This circumstance has been noted many times in the past (e.g. Gidaspo 1974, Bouré 1975). When $V_G = V_L$ the eigenvalue equation [31] has the roots $c = 0$ and $c = (\beta_{GG} + \beta_{LL})/k$. The second root corresponds to a damped perturbation when the stability condition [44] is satisfied, while the first root describes a neutrally stable perturbation propagating with the common velocity of the phases $V_G = V_L = U$.

Since in the framework of this simple model in a steady uniform flow the phases must move with a common velocity U , such a flow is completely specified by U , the pressure gradient $\partial p/\partial x$, and its composition α_G (or α_L). These three quantities are connected by [23] so that the specification of any one of them determines the other two. Thus, for example, a specified pressure gradient fixes the velocity and the composition of the two-phase flow. Due to the nonlinearity of [23] more than one solution may exist, but in any case, for a fixed pressure gradient, the composition of a steady uniform flow cannot be specified arbitrarily. This feature appears to be unphysical since one would expect to be able to specify the pressure gradient and volume fraction independently, the flow velocity then being determined.

AN EQUAL-PRESSURE MODEL WITH OBJECTIVE ADDED MASS TERMS

A form of the added mass interaction between the phases has been developed by Drew *et al.* (1979) exploiting the objectivity principle of continuum mechanics. For the gas phase this

added mass force has the form

$$F_G = C_{VM} \left\{ \frac{\partial}{\partial t} (V_L - V_G) + [(2 - \lambda)V_G + (\lambda - 1)V_L] \cdot \nabla V_L - [\lambda V_L + (1 - \lambda)V_G] \cdot \nabla V_G \right\}, \quad [55]$$

while that for the liquid phase is obtained by interchange of the indices G, L . C_{VM} and λ are, in principle, functions of the flow variables and $0 \leq \lambda \leq 2$. Since added mass forces should vanish in single-phase flow we define a new dimensionless parameter C_M by

$$C_{VM} = \alpha_L \alpha_G \bar{\rho} C_M, \quad [56]$$

where

$$\bar{\rho} = \alpha_L \rho_L + \alpha_G \rho_G$$

is the mixture density. Extensions of the Wallis model which include the terms [55] may be written as

$$\begin{aligned} \frac{\partial}{\partial t} (\alpha_G V_G) + \frac{\partial}{\partial x} (\alpha_G V_G^2) + \frac{\alpha_G}{\rho_G} \frac{\partial p}{\partial x} \\ = \alpha_G A_G + \alpha_G \alpha_L \frac{\bar{\rho}}{\rho_G} C_M \left\{ \frac{\partial V_L}{\partial t} - \frac{\partial V_G}{\partial t} + [(2 - \lambda)V_G + (\lambda - 1)V_L] \frac{\partial V_L}{\partial x} \right. \\ \left. - [\lambda V_L + (1 - \lambda)V_G] \frac{\partial V_G}{\partial x} \right\}, \end{aligned} \quad [57a]$$

$$\begin{aligned} \frac{\partial}{\partial t} (\alpha_L V_L) + \frac{\partial}{\partial x} (\alpha_L V_L^2) + \frac{\alpha_L}{\rho_L} \frac{\partial p}{\partial x} \\ = \alpha_L A_L + \alpha_G \alpha_L \frac{\bar{\rho}}{\rho_L} C_M \left\{ \frac{\partial V_G}{\partial t} - \frac{\partial V_L}{\partial t} - [(2 - \lambda)V_G + (\lambda - 1)V_L] \frac{\partial V_L}{\partial x} \right. \\ \left. + [\lambda V_L + (1 - \lambda)V_G] \frac{\partial V_G}{\partial x} \right\}, \end{aligned} \quad [57b]$$

This form with $\lambda = 1$ corresponds to that used in RELAP-5 (Ransom *et al.* 1980) and RISQUE (Andersen 1977). Comparing [55] with [2] we readily identify q_{ij} and r_{ij} and obtain the following expressions for the quantities appearing in [31]:

$$\eta_L - \eta_G - \eta = \frac{C_M \bar{\rho} \bar{\rho}}{\rho_G \rho_L} = \frac{\alpha_G \rho_G + \alpha_L \rho_L}{\alpha_G \rho_L + \alpha_L \rho_G} C_M, \quad [58]$$

$$\theta_{L,G} = \eta(V_{G,L} + \Lambda), \quad [59]$$

$$K = \frac{2W + \eta(V_G + V_L) + \eta\Lambda}{1 + \eta}, \quad [60]$$

$$F = \frac{K^2}{4} (1 + \eta) - \bar{\rho} \left[\frac{\alpha_G}{\rho_G} V_L^2 + \frac{\alpha_L}{\rho_L} V_G^2 \right] - \eta V_G V_L - \eta \Lambda (\alpha_L V_G + \alpha_G V_L), \quad [61]$$

where

$$\Lambda = (1 - \lambda)(V_G - V_L).$$

The denominator of the second stability condition [44] has the value $1 + \eta$ and, since $C_M \geq 0$, this is positive and therefore it only requires $\beta_{LL} + \beta_{GG} \leq 0$. The first stability condition [47], upon division by $(V_G - V_L)^2$, simplifies to

$$C_M[\beta_{LL}\beta_{GG} + (1 - \lambda)(\beta_{LL} + \beta_{GG})(\alpha_G\beta_{GG} - \alpha_L\beta_{LL})] \geq \frac{\rho_G\rho_L}{\bar{\rho}} \left[\frac{\alpha_L}{\rho_L} \beta_{LL}^2 + \frac{\alpha_G}{\rho_G} \beta_{GG}^2 \right]. \quad [62]$$

Clearly, this condition can be satisfied when the left side is bounded away from zero and positive by choosing C_M large enough. As an example we may consider the drag functions [4], without structural contribution, i.e. with $K_G = K_L = 0$. We consider the following specific form for the interphase drag coefficient H , which is that used in the code RELAP-5 (Ransom *et al.* 1980),

$$H = N\rho_x\alpha_x |V_L - V_G|, \quad [63]$$

where N is a constant and $\alpha_x = \alpha_G, \rho_x = \rho_L$ when $\alpha_G < \alpha_L$, while $\alpha_x = \alpha_L, \rho_x = \rho_G$ for $\alpha_G > \alpha_L$. With [4] and [63] we find for the functions β

$$\beta_{GG} = -3N\bar{\rho} \frac{|V_G - V_L|}{\rho_L\alpha_G} \alpha_L, \quad [64a]$$

$$\beta_{LL} = -N\bar{\rho} \frac{|V_G - V_L|}{\rho_L\alpha_G} (2\alpha_G - \alpha_L), \quad [64b]$$

when $\alpha_L < \alpha_G$ while the appropriate forms in the case $\alpha_L > \alpha_G$ are obtained by interchanging the indices L and G . Upon insertion of these expressions into [62] we obtain stability bounds C_M^* which are illustrated in figures 1 and 2. For $C_M > C_M^*$ a steady uniform flow described by [38] with the drag coefficient [63] is stable. Figure 1 refers to the case $\rho_G/\rho_L = 10^{-3}$ which is typical of gas-liquid flows, while figure 2 refers to the equal density case $\rho_G = \rho_L$ which is typical of liquid-liquid flows. For $\alpha_G < 0.5$ and λ not too close to zero, one finds reasonably low values of C_M^* . For $\alpha_G > 0.5$, low values of λ appear more suitable. The above results are

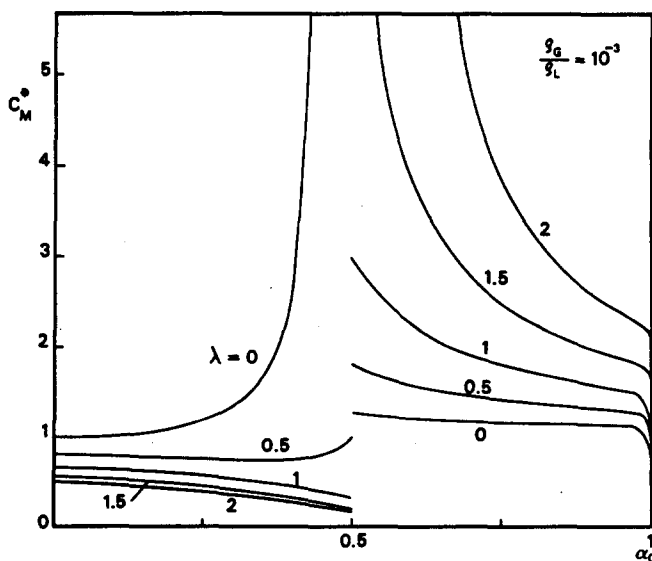


Figure 1. The value of C_M^* is shown for $\rho_G/\rho_L = 10^{-3}$ as a function of α_G for several values of λ according to [62] with the drag function [63]. A value of the added mass coefficient $C_M > C_M^*$ ensures the stability of steady uniform flow.

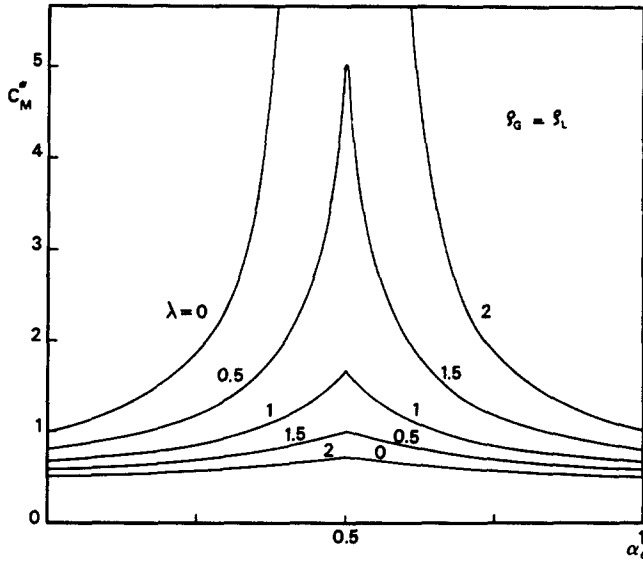


Figure 2. The value of C_M^* is shown for $\rho_G = \rho_L$ as a function of α_G for several values of λ according to [62] with the drag function [63]. A value of the added mass coefficient $C_M > C_M^*$ ensures the stability of steady uniform flow.

presented mainly as examples of one way in which our stability theory can be exploited in the assessment of proposed drag functions.

It is also of interest to examine the condition for reality of characteristics for [57] with added mass terms. The condition $F/A \geq 0$ is found to reduce to

$$\begin{aligned}
 (V_G - V_L)^2 \left\{ \frac{1}{4} \eta^2 [1 + 2(1 - \lambda)(\alpha_G - \alpha_L) + (1 - \lambda)^2] \right. \\
 \left. + \eta \tilde{\rho} (1 - \lambda) \alpha_G \alpha_L \left(\frac{1}{\rho_L} - \frac{1}{\rho_G} \right) - \tilde{\rho}^2 \frac{\alpha_G \alpha_L}{\rho_G \rho_L} \right\} \geq 0.
 \end{aligned}
 \tag{65}$$

If $V_G - V_L \neq 0$ this condition in terms of C_M becomes

$$\begin{aligned}
 C_M^2 [1 + 2(1 - \lambda)(\alpha_G - \alpha_L) + (1 - \lambda)^2] \\
 - 4C_M \frac{\rho_L - \rho_G}{\tilde{\rho}} (1 - \lambda) \alpha_G \alpha_L - 4\alpha_G \alpha_L \frac{\rho_G \rho_L}{\tilde{\rho}^2} \geq 0.
 \end{aligned}
 \tag{66}$$

It may be shown that [66] is satisfied by $C_M \geq 1$ for any λ in the range (0, 2).

It is clear from the above that for C_M sufficiently large stable steady uniform flows with unequal velocities are possible.

A MODEL FOR DISPERSE FLOW

Voinov & Petrov (1979) have derived an alternative expression for the added mass force in disperse flow. In the incompressible case this reduces to

$$F_G = -\frac{1}{2} \rho_L \alpha_G \left(\frac{\partial V_G}{\partial t} + V_G \frac{\partial V_G}{\partial x} - \frac{\partial V_L}{\partial t} - V_L \frac{\partial V_L}{\partial x} \right),
 \tag{67}$$

for the disperse phase (which, for definiteness, we take to be the gas) while the appropriate form for the other phase is $F_L = -F_G$.

It may be noted that the Drew *et al.* (1979) expression does not reduce to this form for any value of the parameter λ so that (67) is not objective. In view of the ambiguous status of

objectivity in current multiphase flow modelling (Drew 1983, Ryskin & Rallison 1980) this circumstance may not appear to justify rejection of [67] out of hand.

If [67] is used in place of [55] in the momentum equations [57] the characteristics of the resulting system are found to be complex (Prosperetti & Van Wijngaarden 1976). However, in addition to the added mass force [67] in a disperse two-phase flow further interactions are present certain of which, as suggested by Prosperetti & Jones (1984) and by Biesheuvel & van Wijngaarden (1984), have the form

$$Q \rho_L \frac{\partial}{\partial x} [\alpha_G (V_G - V_L)^2] \quad [68]$$

where Q is a dimensionless coefficient. According to the authors cited, this term should only appear in the continuous phase (liquid) momentum equation. We are therefore led to consider as our third example the following momentum equations:

$$\begin{aligned} \frac{\partial}{\partial t} (\alpha_G V_G) + \frac{\partial}{\partial x} (\alpha_G V_G^2) + \frac{\alpha_G}{\rho_G} \frac{\partial p}{\partial x} + \alpha_G A_G \\ - \frac{1}{2} \alpha_G \frac{\rho_L}{\rho_G} M \left(\frac{\partial V_G}{\partial t} + V_G \frac{\partial V_G}{\partial x} - \frac{\partial V_L}{\partial t} - V_L \frac{\partial V_L}{\partial x} \right), \end{aligned} \quad [69a]$$

$$\begin{aligned} \frac{\partial}{\partial t} (\alpha_L V_L) + \frac{\partial}{\partial x} (\alpha_L V_L^2) + \frac{\alpha_L}{\rho_L} \frac{\partial p}{\partial x} + \alpha_L A_L \\ - Q \frac{\partial}{\partial x} [\alpha_G (V_G - V_L)^2] + \frac{1}{2} \alpha_G M \left(\frac{\partial V_G}{\partial t} + V_G \frac{\partial V_G}{\partial x} - \frac{\partial V_L}{\partial t} - V_L \frac{\partial V_L}{\partial x} \right), \end{aligned} \quad [69b]$$

The dimensionless positive parameter M multiplying [67] has been introduced for increased generality.

The auxiliary quantities defined in [13]–[18] are readily computed. In particular one finds

$$\eta_G - \eta_L = \frac{M \tilde{\rho}}{2 \rho_G \alpha_L}, \quad [70]$$

$$\theta_G = \frac{\tilde{\rho}}{\rho_G \alpha_L} \left[\frac{1}{2} M V_G + \alpha_G Q (V_G - V_L) \right] \quad [71a]$$

$$\theta_L = \frac{\tilde{\rho}}{\rho_G \alpha_L} \left[\frac{1}{2} M V_L + (1 + \alpha_G) Q (V_G - V_L) \right] \quad [71b]$$

Since $A = 1 + \alpha_G \eta_L + \alpha_L \eta_G$ is positive, the characteristics are real provided that, equation [51]

$$\frac{\alpha_G}{\alpha_L^2} Q^2 + \left[\frac{-M}{2\alpha_L} + \frac{\rho_G}{\rho_L} (1 + \alpha_G) - \alpha_G \right] Q - \frac{\rho_G}{\rho_L} \alpha_L - \frac{M}{2} \left(1 + \frac{\rho_G}{\rho_L} \right) - \frac{M^2}{4\alpha_L} > 0 \quad [72]$$

where a common factor $(V_G - V_L)^2 \tilde{\rho}^2 \alpha_G / \rho_G$ has been omitted. The inequality is satisfied for $Q < Q_-$ and $Q > Q_+$, with Q_{\pm} the two roots of the quadratic.

At low gas volume fraction and for $\rho_G \ll \rho_L$ one finds

$$Q_+ \approx 1 + \frac{1}{2} M + 0(\alpha_G) \quad [73]$$

$$Q \approx -\frac{1}{2} \frac{M}{\alpha_G \alpha_L} + O(1) \quad [74]$$

The second root is seen to diverge in this limit. In the paper by Prosperetti & Jones (1984) the value $1/4$ was suggested for Q , but it was recognised that other contributions would be present in a complete model arising from correlation terms (Reynolds stresses) and other effects. The model of Biesheuvel & van Wijngaarden includes some of these effects and results in a net Q which is negative. In both cases the characteristics are then complex for $\alpha_G \rightarrow 0$. This conclusion indicates that terms having a different structure from [68] should be present in a complete model.

CONCLUSIONS

The purpose of the present study has not been the proposal of a specific two-phase flow model, but rather the examination of a general class of one-dimensional models containing only first-order derivatives and algebraic terms, see [1] and [2]. This class of models is sufficiently general to include several formulations of the added mass force and other interactions between the phases including drag forces and drag forces with a solid structure. Furthermore different pressure gradients are allowed in the two momentum equations, provided that their difference can be expressed as a linear combination of derivatives of the other flow variables, namely velocities and volume fractions.

We have carried out an analysis of the linear stability of steady uniform flow as described by this general class of models and have derived explicit stability criteria. Despite the presence of nondifferential terms in the model, it has been shown that the stability properties do not depend upon the wavelength of the perturbation. This feature is certainly unphysical and indicates that the model considered, in spite of its generality, is incomplete. This is reminiscent of the situation encountered in the stability analysis of tangential velocity discontinuities when gravity and surface tension effects are omitted (Landau & Lifshitz 1959). A valid model would presumably contain higher value space derivatives such as those representing surface-tension effects (Ramshaw & Trapp 1978). Our analysis of general models containing second-order space derivatives (Prosperetti & Jones, 1985) does in fact indicate a wavelength dependence of the stability criteria, although the results of the present paper are recovered for long waves.

Another possible feature of a more realistic model which is not included exactly in the formulation examined here is the explicit appearance of the pressure elsewhere than in the pressure gradient, e.g. in a term like $p \nabla \alpha_G$.† Should terms of this form be present in a steady uniform flow the coefficients of the perturbed equations would depend on the space coordinate through p . In this case our results are approximately valid for perturbations of sufficiently small wavelength, i.e. small compared with the characteristic length of variation of p (Ramshaw & Trapp 1978).

In general, hyperbolicity and stability coincide only as the wavelength of the perturbation tends to zero (Ramshaw & Trapp 1978). However due to the wavelength independence of the stability properties of the present model, we find that hyperbolicity is a necessary condition for stability.

In the light of the general results obtained we have analysed three specific models and we have obtained bounds on the parameters appearing in order for the models to be stable. Another possible use of our results which we have illustrated when considering the second example is in the assessment of interphase drag functions. In addition we have considered

† There are good reasons to believe that the term $p \nabla \alpha_G$ should not be present for disperse two-phase flows (Prosperetti & Jones 1984).

steady nonuniform flow and have obtained an explicit relationship between the velocities of the phases as the flow evolves in space.

At the root of our consideration is the assumption that steady uniform flows are a physical possibility for two-phase flow systems. Even if the flow cannot literally be steady and uniform, if for no other reason than the discrete nature of the phases, by a suitable definition of averaging the flow variables appearing in the conservation equations can be made independent of space and time (except of course, for the pressure) in a number of practical situations.

In order to obtain the results described it has been necessary to assume that the two phases are individually incompressible. This is certainly justified in liquid-liquid and liquid-solid flows in a wide range of conditions, and in some flows involving a gas (e.g. droplet flow) at low speed. With this assumption the energy equations decouple from the other equations and need not be considered.

Continuity arguments suggest that results similar to ours will hold in the compressible case, at least at low velocities. The stability analysis would however have to start from the complete equation system (including the equations of state and possibly the energy equations) rather than from the simplified system [11] obtained by the elimination of the pressure gradient and would be significantly more complex.

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